

CONSTRUCTION OF CONJUGATE FUNCTIONS

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ABSTRACT. We find all pairs of real analytic functions f and g in \mathbb{R}^n such that $|\nabla f| = |\nabla g|$ and $(\nabla f)(\nabla g) = 0$.

We consider the following problem: Let $n \geq 3$. Find all pairs of functions (f, g) , $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $g : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$|\nabla f| = |\nabla g|, \quad (\nabla f)(\nabla g) = 0.$$

Functions f and g constituting such a pair are called *conjugate*.

Finding a pair of conjugate functions (f, g) is equivalent to finding a complex valued function $h : \mathbb{R}^n \rightarrow \mathbb{C}$ such that

$$(\nabla h)(\nabla h) = 0.$$

Having h one gets f and g as its real and imaginary parts, respectively.

Solving the problem at algebraic level, one needs to find a nice representation for *null* vectors in \mathbb{C}^n . This may be done by means of spinors.

The solution of the problem in dimension $n = 3$ is as follows.

We consider \mathbb{R}^3 with coordinates (x, y, z) . The most general form of ∇h is given in terms of two complex-valued functions (a spinor) (ϕ_1, ϕ_2)

$$h_x = \phi_1^2 - \phi_2^2, \quad h_y = i(\phi_1^2 + \phi_2^2), \quad h_z = 2\phi_1\phi_2.$$

Now, the integrability condition for existence of h is

$$d [(\phi_1^2 - \phi_2^2)dx + i(\phi_1^2 + \phi_2^2)dy + 2\phi_1\phi_2dz] = 0,$$

which is equivalent to

$$[\phi_1(dx + idy) + \phi_2dz] \wedge d\phi_1 + [-\phi_2(dx - idy) + \phi_1dz] \wedge d\phi_2 = 0.$$

This motivates introduction of two functions

$$X_1 = \phi_1(x + iy) + \phi_2z, \quad X_2 = -\phi_2(x - iy) + \phi_1z.$$

Having them the integrability condition is:

$$d(X_1d\phi_1 + X_2d\phi_2) = 0.$$

Its general solution analytic in (x, y, z) is

$$X_1 = F_1(\phi_1, \phi_2) \quad X_2 = F_2(\phi_1, \phi_2),$$

where $F = F(\phi_1, \phi_2)$ is a complex valued function analytic in both variables, and $F_1 = \frac{\partial F}{\partial \phi_1}$, $F_2 = \frac{\partial F}{\partial \phi_2}$.

Explicitly, one finds ϕ_1 and ϕ_2 specifying F and solving the algebraic equations

$$\phi_1(x + iy) + \phi_2z = F_1(\phi_1, \phi_2)$$

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$$-\phi_2(x - iy) + \phi_1 z = F_2(\phi_1, \phi_2).$$

Once (ϕ_1, ϕ_2) is found then $h = \int(\phi_1^2 - \phi_2^2)dx$, which solves the problem.

So in dimension $n = 3$ the conclusion is that the general analytic solution is generated by one complex analytic function F of two variables.

Although, for arbitrary n we can not use spinors anymore, a similar procedure works for any $n \geq 3$.

In \mathbb{R}^n we introduce coordinates (x_1, x_2, \dots, x_n) and parametrize a general null vector $(h_{x_1}, h_{x_2}, \dots, h_{x_n})$ by means of $n - 1$ complex functions $(\phi_1, \phi_2, \dots, \phi_{n-1})$ via

$$\begin{aligned} h_{x_1} &= \phi_1^2 + \phi_2^2 + \dots + \phi_{n-2}^2 - \phi_{n-1}^2 \\ h_{x_2} &= i(\phi_1^2 + \phi_2^2 + \dots + \phi_{n-2}^2 + \phi_{n-1}^2) \\ h_{x_3} &= 2\phi_1\phi_{n-1} \\ &\dots \\ h_{x_k} &= 2\phi_{k-2}\phi_{n-1} \\ &\dots \\ h_{x_n} &= 2\phi_{n-2}\phi_{n-1}. \end{aligned}$$

This parametrization is known in the theory of minimal surfaces [4].

Now the integrability condition

$$d[h_{x_1}dx_1 + h_{x_2}dx_2 + \dots + h_{x_n}dx_n] = 0$$

for existence of $h : \mathbb{R}^n \rightarrow \mathbb{C}$ is

$$d[X_1d\phi_1 + X_2d\phi_2 + \dots + X_{n-1}d\phi_{n-1}] = 0$$

with $n - 1$ functions $(X_1, X_2, \dots, X_{n-1})$ given by

$$\begin{aligned} X_1 &= \phi_1(x_1 + ix_2) + \phi_{n-1}x_3 \\ X_2 &= \phi_2(x_1 + ix_2) + \phi_{n-1}x_4 \\ &\dots \\ X_{n-2} &= \phi_{n-2}(x_1 + ix_2) + \phi_{n-1}x_n \\ X_{n-1} &= -\phi_{n-1}(x_1 - ix_2) + \phi_1x_3 + \phi_2x_4 + \dots + \phi_{n-2}x_n. \end{aligned}$$

Thus, similarly to the $n = 3$ case, we first choose an arbitrary holomorphic function $F = F(\phi_1, \phi_2, \dots, \phi_{n-1})$ of $n - 1$ complex variables and, denoting its derivatives by $F_i = \frac{\partial F}{\partial \phi_i}$, $i = 1, 2, \dots, n - 1$, solve algebraic equations

$$\begin{aligned} F_1 &= \phi_1(x_1 + ix_2) + \phi_{n-1}x_3 \\ F_2 &= \phi_2(x_1 + ix_2) + \phi_{n-1}x_4 \\ &\dots \\ F_{n-2} &= \phi_{n-2}(x_1 + ix_2) + \phi_{n-1}x_n \\ F_{n-1} &= -\phi_{n-1}(x_1 - ix_2) + \phi_1x_3 + \phi_2x_4 + \dots + \phi_{n-2}x_n, \end{aligned}$$

for $(\phi_1, \phi_2, \dots, \phi_{n-1})$ as functions of (x_1, x_2, \dots, x_n) . Then h is found by simple integration to be e.g. $h = \int 2\phi_{n-2}\phi_{n-1}dx_n$.

We note that, under some regularity assumptions on the function F , equations (0.1) associate to any solution h an n -dimensional real hypersurface M_n embedded in \mathbb{C}^{n-1} . One gets the equations for this hypersurface in coordinates $(\phi_1, \phi_2, \dots, \phi_{n-1}) \in \mathbb{C}^{n-1}$ by eliminating the real parameters (x_1, x_2, \dots, x_n) from

equations (0.1). For example if $n = 3$, given $F = F(\phi_1, \phi_2)$, we have a 3-dimensional real hypersurface M_3 in \mathbb{C}^2 defined by

$$M_3 = \{ (\phi_1, \phi_2) \in \mathbb{C}^2 : \text{Im}(F_1 \bar{\phi}_2 - F_2 \bar{\phi}_1) = 0 \}.$$

In case of $n = 4$, given $F = F(\phi_1, \phi_2, \phi_3)$ we have

$$\begin{aligned} M_4 &= \{ (\phi_1, \phi_2, \phi_3) \in \mathbb{C}^3 : \\ 0 &= \text{Im} \left(\bar{\phi}_3 [F_1(\phi_2^2 \bar{\phi}_1^2 + |\phi_2|^4 - |\phi_1 \phi_3|^2 - |\phi_3|^4) - \right. \\ &\quad \left. F_2(|\phi_2|^2 \phi_1 \bar{\phi}_2 + (|\phi_1|^2 + |\phi_3|^2) \phi_2 \bar{\phi}_1) + F_3 \phi_3(\phi_1 \bar{\phi}_2^2 + (|\phi_1|^2 + |\phi_3|^2) \bar{\phi}_1)] \right) \\ 0 &= \text{Im} \left(\bar{\phi}_3 [F_1(|\phi_1|^2 \phi_2 \bar{\phi}_1 + (|\phi_2|^2 + |\phi_3|^2) \phi_1 \bar{\phi}_2) - \right. \\ &\quad \left. F_2(\phi_1^2 \bar{\phi}_2^2 + |\phi_1|^4 - |\phi_2 \phi_3|^2 - |\phi_3|^4) - F_3 \phi_3(\phi_2 \bar{\phi}_1^2 + (|\phi_2|^2 + |\phi_3|^2) \bar{\phi}_2)] \right) \}. \end{aligned}$$

We also note that hypersurfaces M_n are foliated by $(n-2)$ -dimensional leaves which are the images under the map $(x_1, x_2, \dots, x_n) \mapsto (\phi_1, \phi_2, \dots, \phi_{n-1})$ of the intersections in \mathbb{R}^n of the level surfaces $f = c_1$ and $g = c_2$ corresponding to the conjugate functions (f, g) associated with h . Thus, in particular, M_3 has a distinguished foliation by real curves and M_4 has a distinguished foliation by real surfaces.

Returning to our original problem it is interesting to note that slightly different procedure, based on a *three*-linear representation of a complex null vector, can be used to solve the problem in dimension $n = 5$.

In \mathbb{R}^5 we use coordinates (x, y, z, t, u) . Now we write a null vector $(h_x, h_y, h_z, h_t, h_u)$ in terms of *six* functions $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6)$ as follows:

$$\begin{aligned} h_x &= i(\phi_1 \phi_2 + \phi_3 \phi_4) \phi_5 - \frac{1}{2}(\phi_1^2 + \phi_2^2 - \phi_3^2 - \phi_4^2) \phi_6, \\ h_y &= i(-\phi_1 \phi_3 + \phi_2 \phi_4) \phi_5 + (\phi_2 \phi_3 - \phi_1 \phi_4) \phi_6 \\ h_z &= (-\phi_1 \phi_2 + \phi_3 \phi_4) \phi_5 - \frac{i}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) \phi_6 \\ h_t &= (\phi_1 \phi_3 + \phi_2 \phi_4) \phi_5 \\ h_u &= (\phi_1 \phi_3 - \phi_2 \phi_4) \phi_6. \end{aligned}$$

Check that $h_x^2 + h_y^2 + h_z^2 + h_t^2 + h_u^2 = 0$. Remarkably again, the integrability conditions

$$d[h_x dx + h_y dy + h_z dz + h_t dt + h_u du] = 0$$

are equivalent to

$$d(Xd\phi_1 + Yd\phi_2 + Zd\phi_3 + Td\phi_4 + Ud\phi_5 + Wd\phi_6) = 0,$$

where

$$\begin{aligned} X &= \phi_3 \phi_5(t - iy) - (\phi_1 \phi_6 - i\phi_2 \phi_5)(x + iz) - \phi_4 \phi_6 y + \phi_3 \phi_6 u \\ Y &= \phi_4 \phi_5(t + iy) - (\phi_2 \phi_6 - i\phi_1 \phi_5)(x + iz) + \phi_3 \phi_6 y + \phi_4 \phi_6 u \\ Z &= \phi_1 \phi_5(t - iy) + (\phi_3 \phi_6 + i\phi_4 \phi_5)(x - iz) + \phi_2 \phi_6 y + \phi_1 \phi_6 u \\ T &= \phi_2 \phi_5(t + iy) + (\phi_4 \phi_6 + i\phi_3 \phi_5)(x - iz) - \phi_1 \phi_6 y + \phi_2 \phi_6 u \\ U &= \phi_1 \phi_3(t - iy) + \phi_2 \phi_4(t + iy) + i\phi_1 \phi_2(x + iz) + i\phi_3 \phi_4(x - iz) \\ W &= -\frac{1}{2}(\phi_1^2 + \phi_2^2)(x + iz) + \frac{1}{2}(\phi_3^2 + \phi_4^2)(x - iz) + (\phi_2 \phi_3 - \phi_1 \phi_4)y + (\phi_1 \phi_3 + \phi_2 \phi_4)u. \end{aligned}$$

Thus taking an analytic function $F = F(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6)$ of six variables, and solving algebraic equations

$$\begin{aligned} F_1 &= \phi_3 \phi_5(t - iy) - (\phi_1 \phi_6 - i\phi_2 \phi_5)(x + iz) - \phi_4 \phi_6 y + \phi_3 \phi_6 u \\ F_2 &= \phi_4 \phi_5(t + iy) - (\phi_2 \phi_6 - i\phi_1 \phi_5)(x + iz) + \phi_3 \phi_6 y + \phi_4 \phi_6 u \\ F_3 &= \phi_1 \phi_5(t - iy) + (\phi_3 \phi_6 + i\phi_4 \phi_5)(x - iz) + \phi_2 \phi_6 y + \phi_1 \phi_6 u \end{aligned}$$

$$\begin{aligned}
F_4 &= \phi_2\phi_5(t+iy) + (\phi_4\phi_6 + i\phi_3\phi_5)(x-iz) - \phi_1\phi_6y + \phi_2\phi_6u \\
F_5 &= \phi_1\phi_3(t-iy) + \phi_2\phi_4(t+iy) + i\phi_1\phi_2(x+iz) + i\phi_3\phi_4(x-iz) \\
F_6 &= -\frac{1}{2}(\phi_1^2 + \phi_2^2)(x+iz) + \frac{1}{2}(\phi_3^2 + \phi_4^2)(x-iz) + (\phi_2\phi_3 - \phi_1\phi_4)y + (\phi_1\phi_3 + \phi_2\phi_4)u
\end{aligned}$$

for $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6)$ as functions of (x, y, z, t, u) , we generate a solution to $(\nabla h)(\nabla h) = 0$ via, for example, an integral $h = \int[(\phi_1\phi_3 + \phi_2\phi_4)\phi_6]du$.

This note is motivated by a paper [1]. It reminds very much the Kerr theorem (see e.g. [2, 3]).

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